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On the number of isolating integrals in perturbed Hamiltonian systems with $n \geqslant 3$ degrees of freedom

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Abstract. We consider a perturbed Hamiltonian system with $n \geqslant 3$ degrees of freedom of the form $H = H_0 + \varepsilon H_1$ and show that the properties of the average value of the perturbing function H_1 along the periodic orbits of the unperturbed integrable part H_0 supply criteria for non-integrability which restrict the allowed total number of independent integrals of motion.

1. Introduction

Only a few criteria for proving non-integrability of Hamiltonian systems of three or more degrees of freedom have appeared in the literature. Ziglin's theorem [1] has been proved for the n-dimensional case but it has found many applications in systems with two degrees of freedom (e.g. [2, 3]), its applications to more than two dimensions having been made after the system has been reduced to two dimensions by the use of known integrals of motion (e.g. [4–6]). Yoshida [7] has proved a non-integrability theorem applicable to n-degrees-of-freedom Hamiltonians which states that, if the differences between pairs of Kowalevski exponents ρ_i, ρ_{i+n} on a straight-line solution of the system are rationally independent, then the Hamiltonian cannot possess analytic integrals of motion other than H itself.

In a recent paper [8], we offered a criterion for the non-integrability of systems with two degrees of freedom which was based on a well known theorem by Poincaré ([9], p 233). In the present paper, by working along the same lines, we prove a theorem on the non-integrability of nearly integrable Hamiltonian systems of n degrees of freedom with $n \geqslant 3$ of the form $H = H_0 + \varepsilon H_1$, where H_0 is a non-degenerate integrable part. We actually show that some properties of the average value of the perturbing function H_1 on the periodic orbits of the integrable part are strongly connected with the integrability of the perturbed system. This average value can be considered as a scalar function on the quotient manifold of a resonant torus of H_0 with respect to the foliation induced by the periodic orbits of H_0. If the gradient of this function is not identically zero for a dense set of resonant tori of H_0, the perturbed system cannot possess a complete set of integrals, analytic in ε, and thus it is non-integrable for an open interval of ε around zero. On the other hand, if the rank of the Hessian determinant of this function equals r then the perturbed system cannot possess more than $n - r$ independent integrals, including H, which are analytic in ε around $\varepsilon = 0$.

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2. Non-existence of integrals in n-degrees-of-freedom Hamiltonians

Consider the n-degrees-of-freedom nearly integrable Hamiltonian of the form

$$H = H_0 + \varepsilon H_1 \tag{1}$$

where H_0 is integrable, i.e. it possesses $n-1$ (in addition to H_0) single-valued independent integrals in involution, and suppose that we can define action-angle variables $J_i, w_i (i = 1, .., n)$, at least in an open domain of phase space. We also suppose that H_0 is non-degenerate, i.e.

$$\det\left[\frac{\partial^2 H_0}{\partial J_i \partial J_j}\right] \neq 0 \tag{2}$$

a fact we will use later on.

In order to obtain necessary conditions for the integrability of H, we assume that it possesses n independent integrals $H, \Phi_1, \ldots, \Phi_{(n-1)}$, analytic in an open interval around $\varepsilon = 0$, i.e. Φ_i are expandable as

$$\Phi_i = \Phi_{i0} + \varepsilon\Phi_{i1} + O(\varepsilon^2) \qquad (i = 1, \ldots, n-1). \tag{3}$$

By using the involution property $[H, \Phi_i] = 0$ $(i = 1, \ldots, n-1)$ and equating terms of the same order in ε, we obtain

$$[H_0, \Phi_{i0}] = 0 \tag{4}$$

$$[H_0, \Phi_{i1}] + [H_1, \Phi_{i0}] = 0. \tag{5}$$

Equations (4) and (5) have been derived by Poincaré ([9], p 233) and hold identically in phase space. Equations (4) indicate that Φ_{i0} are integrals of H_0. It is known [9] that if H is integrable, Φ_i can always be selected such that $H_0, \Phi_{10}, \ldots, \Phi_{(n-1)0}$ are independent. This is an important conclusion since it will provide us with the contradiction needed to prove our main result. Another important property of Φ_{i0} stems from the non-degeneracy condition (2), i.e. Φ_{i0} do not depend on the angles [9]. This holds for the zeroth-order terms of whatever isolating integrals of the perturbed Hamiltonian as long as the unperturbed part H_0 obeys the non-degeneracy condition. Since we are interested in disproving the existence of all possible isolating integrals, we will only use these two properties and relations (4) and (5) which are necessary conditions for all possible independent integrals Φ_i of H.

Parametrizing equations (5) along the orbits of the unperturbed motion, we derive that

$$\frac{d\Phi_{i1}}{dt} = [H_1, \Phi_{i0}] \tag{6}$$

where both sides are evaluated along a particular solution of H_0.

On any torus of H_0, the unperturbed solution is described by the angle coordinates as follows:

$$w_i = \omega_i t + \vartheta_i \qquad (i = 1, \ldots, n) \tag{7}$$

where the w_i are mod(2π) and the ϑ_i are arbitrary initial conditions on the torus. In what follows, we will concentrate on the periodic motions of a resonant torus of H_0 with frequency ratio

$$\frac{\omega_1}{p_1} = \frac{\omega_2}{p_2} = \cdots = \frac{\omega_n}{p_n} \tag{8}$$

where $\omega_i = \partial H_0/\partial J_i$ are the frequencies of the integrable part and p_i are non-zero integers with no common divisor. A particular resonant torus is foliated by periodic orbits with period

$$T = \frac{2\pi p_i}{\omega_i}. \tag{9}$$

Every periodic orbit is mapped on the quotient manifold of the resonant torus with respect to the foliation induced by the trivially integrable 1-forms $p_{i+1}\,\mathrm{d}w_1 - p_1\,\mathrm{d}w_{i+1} = 0$ $(i = 1, \dots, n-1)$, to a point with coordinates

$$\varphi_i = p_{i+1}\vartheta_1 - p_1\vartheta_{i+1} \bmod(2\pi s_i) \qquad (i = 1, \dots, n-1) \tag{10}$$

where the s_i are the common divisors of the pairs p_1, p_{i+1}.

By integrating equations (6) along any periodic orbit on the torus and demanding that Φ_{i1} be single-valued functions, we deduce the equations

$$\int_0^T [H_1, \Phi_{i0}]\,\mathrm{d}t = 0 \qquad (i = 1, \dots, n-1)$$

which must hold on every solution on this resonant torus. Since Φ_{i0} are unknown, we need to investigate further the above equations, which can be written in the form

$$\sum_{j=1}^{n} \frac{\partial \Phi_{i0}}{\partial J_j} \int_0^T \frac{\partial H_1}{\partial w_j}\,\mathrm{d}t = 0. \tag{11}$$

By virtue of the equations of motion (7), equations (11) transform to

$$\sum_{j=1}^{n} \frac{\partial \Phi_{i0}}{\partial J_j} \frac{\partial}{\partial \vartheta_j} \langle H_1 \rangle = 0 \tag{12}$$

where $\langle H_1 \rangle$ is the average value of H_1 along the particular periodic orbit and depends on the initial conditions ϑ_i only through the parameters φ_j. Thus, if we take into account relations (10), equations (12) become

$$\sum_{j=1}^{n-1} D_j^{(i)} \frac{\partial \langle H_1 \rangle}{\partial \varphi_j} = 0 \qquad (i = 1, \dots, n-1) \tag{13}$$

where

$$D_j^{(i)}(J_1, \dots, J_n) = p_{j+1}\frac{\partial \Phi_{i0}}{\partial J_1} - p_1 \frac{\partial \Phi_{i0}}{\partial J_{j+1}}. \tag{14}$$

For every integral Φ_{i0} of H_0 there exists a corresponding quantity ${D_j}^{(i)}$ defined by equation (14). Suppose now that s integrals $\Phi_{k0} (k = 1, \dots, s)$ produce linearly dependent vectors ${D_j}^{(k)}$, i.e.

$$\sum_{k=1}^{s} \lambda_k D_j^{(k)} = 0 \qquad (j = 1, \dots, n-1). \tag{15}$$

Then, the gradients of the integrals $\Phi_{10}, \ldots, \Phi_{s0}, H_0$ are also linearly dependent, i.e.

$$\sum_{k=1}^{s} \mu_k \frac{\partial \Phi_{k0}}{\partial J_r} + \mu_0 \frac{\partial H_0}{\partial J_r} = 0 \qquad (r = 1, \ldots, n) \tag{16}$$

with

$$\mu_k = -\lambda_k \frac{\partial H_0}{\partial J_1} \qquad \mu_0 = \sum_{k=1}^{s} \lambda_k \frac{\partial \Phi_{k0}}{\partial J_1}.$$

If, on at least one orbit, the vector $\partial \langle H_1 \rangle / \partial \varphi_j$ is non-zero, then, from equation (13), we get

$$\det |D_j^{(i)}| = 0$$

and, as shown above, this leads to

$$\frac{D(H_0, \Phi_{10}, \ldots, \Phi_{(n-1)0})}{D(J_1, J_2, \ldots, J_n)} = 0. \tag{18}$$

If the vector $\partial \langle H_1 \rangle / \partial \varphi_j$ is different from zero on a dense set of resonant tori of H_0, on at least one orbit on each torus, equation (18) contradicts the fact that Φ_i are independent integrals of H, i.e. H does not possess a complete set of integrals, analytic in an open interval around zero.

In what follows, we will obtain conditions for the non-existence of a certain number $s < n - 1$ of independent integrals Φ_i of H. Every such integral will supply a relation of the form (13). By differentiating this relation with respect to φ_m, we form the homogeneous linear system

$$\sum_{j=1}^{n-1} \frac{\partial^2 \langle H_1 \rangle}{\partial \varphi_m \partial \varphi_j} D_j^{(i)} = 0. \tag{19}$$

The matrix in (19) depends on the particular periodic orbit, while $D_j{}^{(i)}$, according to (14), depend only on the selected resonant torus. The maximum allowable number of independent non-zero solutions of (19) equals $(n - 1) - \mathrm{rank}(M)$, where M is the Hessian matrix in (19). Let us suppose that on the set of periodic orbits of this particular torus

$$\max \left\{ \mathrm{rank} \left| \frac{\partial^2 \langle H_1 \rangle}{\partial \varphi_m \partial \varphi_j} \right| \right\} = r_i$$

where the subscript i denotes the particular torus. This means that there exists at least one orbit with $\mathrm{rank}(M) = r_i$ and, therefore, the least number of independent solutions of (19) is allowed. Then system (19) admits, at most, $(n - 1 - r_i)$ independent non-zero solutions. On the other hand, equations (15) and (16) show that, if H possesses s independent integrals, in addition to H, then equation (19), evaluated on any periodic orbit, possesses s independent non-zero solutions of the form (14), which means that

$$s \leqslant n - 1 - r_i$$

and thus Hamiltonian H cannot possess more than $(n - r_i)$ independent integrals, including H, on this particular resonant torus. In order to find the maximum number of allowed independent integrals in the open domain of phase space where action-angle variables are defined, we need to consider the rank of M on a dense set of resonant tori in this domain. If, on this dense set,

$$\min r_i = r$$

then it is certain that $s \leqslant n - 1 - r$, since, for each of these tori, $n - 1 - r_i \leqslant n - 1 - r$. Thus, we may state the following theorem.

Theorem. Consider the n-degrees-of-freedom Hamiltonian $H \doteq H_0 + \varepsilon H_1$, where H_0 is a non-degenerate integrable part for which action-angle variables can be defined in an open domain of phase space. The quotient manifold $Q(J_1, \ldots, J_n)$ of each resonant torus of H_0 with respect to the foliation induced by the periodic orbits is the $(n-1)$-dimensional torus T^{n-1}. Let $\langle H_1 \rangle$ be the average value of H_1 along these periodic orbits which is a scalar function on each Q. Then,

(i) if in this open domain of phase space, a dense set of resonant tori of H_0 is found such that for at least one orbit on each one of them

$$\frac{\partial}{\partial \varphi_i} \langle H_1 \rangle \neq 0 \tag{20}$$

for at least one i, then H does not possess a complete set of analytic integrals for $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$; and

(ii) if, similarly, on a dense set of resonant tori

$$\operatorname{rank} \left| \frac{\partial^2 \langle H_1 \rangle}{\partial \varphi_m \partial \varphi_j} \right| \geqslant r \qquad (m, j = 1, \ldots, n-1) \tag{21}$$

then H cannot possess more than $(n - r)$ independent integrals, including H, which are analytic in the same interval of ε.

3. Application to the three-dimensional quartic oscillator

We will apply the previous results to the system of three quartic oscillators, with a weak coupling described by the Hamiltonian

$$H = H_0 + \varepsilon H_1 = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \tfrac{1}{4}(x_1^4 + x_2^4 + x_3^4) + \varepsilon(x_1 x_2 + x_1 x_3) \tag{22}$$

where $P_i = \mathrm{d}x_i/\mathrm{d}t$, and prove that H does not possess another integral of motion $\forall \varepsilon \neq 0$.

The solution of the integrable part is ([10], p 207)

$$x_i = \lambda_i \operatorname{cn}\left(\lambda_i t - \vartheta_i, \frac{1}{\sqrt{2}}\right) \qquad (i = 1, 2, 3) \tag{23}$$

where λ_i relate to the actions and define an invariant torus, while ϑ_i determine the initial point of the orbit on the torus. A resonant torus of H_0 is defined by the parameter

$$g = \frac{\lambda_1}{p_1} = \frac{\lambda_2}{p_2} = \frac{\lambda_3}{p_3}$$

where p_i are non-zero integers with no common divisor. The period of the orbits on this torus is $T = 4K/g$, where $K = K(1/\sqrt{2})$ is the complete elliptic integral of the first kind with modulus $k = 1/\sqrt{2}$. The solution (23) on this torus takes the form

$$x_i = g p_i \operatorname{cn}(g p_i t - \vartheta_i) \qquad (i = 1, 2, 3)$$

where we have omitted the modulus for simplicity. Let s_1, s_2 be the common divisors of the pairs p_1, p_2 and p_1, p_3, respectively. In particular, let

$$p_1 = s_1 r_{12} = s_2 r_{13} \qquad p_2 = s_1 r_2 \qquad p_3 = s_2 r_3$$

where the pairs r_{12}, r_2 and r_{13}, r_3 are relative primes. The parameters φ_1, φ_2 are given by

$$\varphi_{i-1} = \frac{\pi}{2K}(p_i\vartheta_1 - p_1\vartheta_i) \qquad (i = 2, 3).$$

The average value of the perturbing function $H_1 = x_1x_2 + x_1x_3$ may be determined by using the Fourier expansion of the elliptic cosine ([10], p 168), i.e.

$$x_i = \frac{2\pi g p_i}{kK}\sum_{m_i=0}^{\infty}\frac{q^{m_i+1/2}}{1+q^{2m_i+1}}\cos\left[(2m_i+1)\frac{\pi}{2K}(gp_it - \vartheta_i)\right] \qquad (i = 1, 2, 3) \tag{24}$$

where $q = \mathrm{e}^{-\pi}$ is the elliptic nome. The average values $\langle x_1x_2\rangle$, $\langle x_1x_3\rangle$ become

$$\langle x_1x_i\rangle = \frac{4\pi^2g^2p_1p_i}{K^2}\sum_{m_1=0}^{\infty}\sum_{m_i=0}^{\infty}\frac{q^{m_1+m_i+1}}{(1+q^{2m_1+1})(1+q^{2m_i+1})}$$
$$\times\cos\left\{\frac{\pi}{2K}[(2m_1+1)\vartheta_1 - (2m_i+1)\vartheta_i]\right\}\delta_{p_1(2m_1+1),p_i(2m_i+1)}$$

for $i = 2, 3$, where $\delta_{i,j}$ is the Kronecker delta. From the above expression, we conclude that $\langle x_1x_i\rangle$ is zero unless

$$r_{1i}(2m_1+1) = r_i(2m_i+1).$$

We select the integers p_i to be positive and odd and work on the dense set of resonant tori of H_0 defined by this selection. Since the integers r_{12}, r_2 and r_{13}, r_3 are relative primes, $\langle x_1x_i\rangle$ is non-zero only for values of m_1, m_2, m_3 satisfying

$$2m_1+1 = (2j+1)r_i \qquad 2m_i+1 = (2j+1)r_{1i} \qquad (i = 2, 3)$$

for some integer j. So $\langle H_1\rangle$ obtains the form

$$\langle H_1\rangle = A_2(\varphi_1) + A_3(\varphi_2)$$

where

$$A_i = \langle x_1x_i\rangle = \frac{4\pi^2g^2p_1p_i}{K^2}\sum_{j=0}^{\infty}\frac{q^{(j+1/2)(r_{1i}+r_i)}}{(1+q^{(2j+1)r_{1i}})(1+q^{(2j+1)r_i})}\cos\left[\frac{2j+1}{s_{i-1}}\varphi_{i-1}\right]$$

for $i = 2, 3$. Note that for the selection $\varphi_{i-1} = \pi s_{i-1}/2$, A_i is zero while for $\varphi_{i-1} = 0$, $A_i \neq 0$. This implies that the gradient $\partial\langle H_1\rangle/\partial\varphi_j$ is not identically zero on the dense set of resonant tori, which proves non-integrability of H for $\varepsilon \neq 0$ in an open interval around zero.

On the other hand

$$\det\left|\frac{\partial^2\langle H_1\rangle}{\partial\varphi_i\partial\varphi_j}\right| = \frac{\mathrm{d}^2A_2}{\mathrm{d}\varphi_1^2}\frac{\mathrm{d}^2A_3}{\mathrm{d}\varphi_2^2}$$

where

$$\frac{\mathrm{d}^2A_i}{\mathrm{d}\varphi_{i-1}^2} = -\frac{4\pi^2g^2r_{1i}r_i}{K^2}\sum_{j=0}^{\infty}\frac{(2j+1)^2q^{(j+1/2)(r_{1i}+r_i)}}{(1+q^{(2j+1)r_{1i}})(1+q^{(2j+1)r_i})}\cos\left[\frac{(2j+1)}{s_{i-1}}\varphi_{i-1}\right]$$

which is different from zero, e.g. for $\varphi_1 = \varphi_2 = 0$, and this proves that the Hamiltonian H does not possess a second integral of motion, analytic in an open interval of ε around zero. These results, however, are valid for all ε since the equations of motion are invariant with respect to the transformation

$$x_i \to cx_i \qquad t \to c^{-1}t \qquad \varepsilon \to c^2\varepsilon$$

with arbitrary c. If we select $H_1 = x_1x_2$, we can still prove non-integrability. The Hessian, however, is identically zero and this is due to the existence of a second integral (i.e. the energy along the x_3-axis).

4. Conclusions

Non-integrability properties of a perturbed Hamiltonian of $n \geqslant 3$ degrees of freedom of the form (1) are investigated by studying the average value of the perturbing function H_1 along the non-isolated periodic orbits of the integrable part H_0.

In [8] we have commented on the relationship between the criterion for non-integrability for two-degrees-of-freedom Hamiltonians developed there and the well known theorem of Poincaré ([9], p 233). Part (i) of the theorem of the present paper is a generalization of the main result obtained in [8] and, therefore, relates to Poincaré's theorem in a similar manner. On the other hand, part (ii) is derived only in the case of more than two-degrees-of-freedom autonomous Hamiltonians.

Poincaré [9], instead of considering the dynamical properties incorporated in equation (5), expanded functions H_1, Φ_{i1} in multiple Fourier series with respect to the angles and obtained

$$B_{\bar{s}}\left(\bar{s} \cdot \frac{\partial \Phi_{i0}}{\partial J}\right) = C_{\bar{s}}(\bar{s} \cdot \bar{\omega}) \qquad (i = 1, \ldots, n-1)$$

for every integer vector $\bar{s} = (s_1, s_2, \ldots, s_n) \in Z^n$, where $B_{\bar{s}}, C_{\bar{s}}$ are the coefficients of the multiple Fourier series of H_1 and Φ_{i1}, respectively. Thus, in order to prove non-integrability, one has to check whether $B_{\bar{s}}$ is different from zero at the dense set of points of phase space where

$$\bar{s} \cdot \bar{\omega} = 0 \tag{25}$$

i.e.

$$\bar{s} \cdot \frac{\partial \Phi_{i0}}{\partial J} = 0$$

for every Φ_{i0} which means that the integrals $\Phi_{10}, \Phi_{20}, \ldots, H_0$ are dependent. This leads to consideration of an infinite number of $B_{\bar{s}}$ at a dense set of points and can be applied as a non-integrability test only if one knows the transformation to action-angle variables and the expression for all the coefficients of the multiple Fourier expansion of H_1; or at least whether they vanish or not in the particular set.

Poincaré also dealt with the non-existence of a certain number of integrals for the perturbed Hamiltonian and he considered a dense set of planes defined by (25) in action space and showed that if one can find m independent directions of integer vectors $\bar{s}$, for which the corresponding $B_{\bar{s}}$ do not vanish, then the perturbed system cannot possess more than $(n-1-m)$ integrals of motion independent of H. On the other hand, in order to apply the present theorem, one needs to compute the time average of H_1 along the periodic orbits of H_0 and its derivatives with respect to the initial conditions φ_i which define a particular orbit on a resonant torus.

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